

A FINITENESS PROPERTY OF TORUS INVARIANTS

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ABSTRACT. In this paper the invariant subring R_n of an algebraic torus $T = (\mathbb{C}^\times)^r$ acting on the multi-homogenous polynomial ring

$$S^{\boxtimes n} = \bigoplus_{d=0}^{\infty} (S^{(d)})^{\otimes n},$$

where $S^{(d)}$ is the d th graded piece of the polynomial ring $S = \mathbb{C}[x_1, \dots, x_k]$, is studied from the viewpoint of matrices whose entries sum to zero. Using these weight matrices we prove that there exists a d_1 such that for all positive integers n , the relations of the invariant subring R_n are generated in multi-homogenous degree $\leq d_1$.

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1. INTRODUCTION

Let G be a reductive group and S a finitely generated graded algebra over \mathbb{C} concentrated in non-negative degrees on which G acts. For all positive integers n let

$$R_n = \bigoplus_{k=0}^{\infty} R_n^{(k)}, \quad \text{where } R_n^{(k)} = ((S^{(k)})^{\otimes n})^G,$$

where $S^{(k)}$ is the k th graded piece of S . Note that R_n is a \mathbb{C} -algebra and so is finitely generated. The following uniformity conjecture for the sequence of algebras $\{R_n\}$ was made by Andrew Snowden in an unpublished paper [3]:

Conjecture 1.1 (Snowden). *Keep the above notation.*

- (a) *Generators* — *There exists a positive integer d such that each algebra R_n is generated in degrees $\leq d$.*
- (b) *Relations* — *Let d be as above. For all positive integers n let*

$$P_n = \text{Sym} \left(\bigoplus_{k=0}^d R_n^{(k)} \right).$$

There exists a positive integer s such that the kernel of the surjection $P_n \rightarrow R_n$ is generated in degrees $\leq s$ as a P_n -module for all n .

- (c) *Syzygies* — *There exist integers s_0, s_1, \dots such that for each n there exists a resolution*

$$\cdots \rightarrow F_{n,2} \rightarrow F_{n,1} \rightarrow F_{n,0} \rightarrow R_n \rightarrow 0$$

of R_n by finite P_n -modules $F_{n,i}$ such that $F_{n,i}$ is generated in degrees $\leq s_i$.

Jonathan Brito proved in [1] that Conjecture 1.1(a) is true when G is a complex r -torus $T = (\mathbb{C}^\times)^r$ and S is a polynomial ring $\text{Sym } V = \mathbb{C}[x_1, \dots, x_k]$ where V is a k dimensional polynomial representation of T . The purpose of this paper is to

prove Conjecture 1.1(b) for this specific case. A formal statement of this result is given in Theorem 2.2 (see page 3).

2. TORUS INVARIANTS

Multi-homogeneous monomials. Fix a complex r -torus $T = (\mathbb{C}^\times)^r$ and k -dimensional polynomial representation V of T . Suppose that the representation V is given by the weight vectors $w_1, \dots, w_k \in \mathbb{Z}^r$. We will denote the set of these weights by \mathcal{X} . Then for any positive integer n , the action of T on V naturally extends to an action of T on the *multi-homogeneous polynomial ring*

$$S^{\boxtimes n} = \bigoplus_{d=0}^{\infty} (\text{Sym}^d V)^{\otimes n}$$

in the following way: Suppose that $\mathbb{V} = \{x_1, \dots, x_k\}$ is a basis of V such that for each i ,

$$t \cdot x_i = t^{w_i} x_i = t_1^{w_{i1}} \cdots t_r^{w_{ir}} x_i.$$

For simplicity we will write $|x_i|$ rather than w_i . A pure tensor in $S^{\boxtimes n}$ is a *multi-homogeneous monomial of degree d* if each tensor factor is a monomial in \mathbb{V} of degree d . For every multi-homogeneous monomial

$$f = f_1 \otimes \cdots \otimes f_n, \quad \text{where } f_i = \phi_{i1} \cdots \phi_{id} \quad (1)$$

of degree d let

$$t \cdot f = t^{|\phi_{11}| + \cdots + |\phi_{nd}|} f. \quad (2)$$

The sum in (2) is very unorganized. For example, it remains the same if you permute the tensor products or if you swap out one basis element for another with an equal weight. The notion of weight matrices given below in Definition 2.1 helps organize this sum in a more structured manner.

Definition 2.1. Suppose f is a multi-homogeneous monomial of degree d

$$f = f_1 \otimes \cdots \otimes f_n, \quad \text{where } f_i = \phi_{i1} \cdots \phi_{id}.$$

Then let

$$\|f\| = \begin{bmatrix} |\phi_{11}| & \cdots & |\phi_{1d}| \\ \vdots & & \vdots \\ |\phi_{n1}| & \cdots & |\phi_{nd}| \end{bmatrix}.$$

We will call $\|f\|$ the *weight matrix* associated to f .

Remark. Note that if two or more basis vectors have the same weight vector, then a single weight matrix can be associated to more than one multi-homogeneous monomials. Furthermore, since multiplication is commutative inside the tensor factors, we can permute the entries in any fixed row of $\|f\|$ and the resulting matrix would still be a weight matrix associated with f . Since we don't want this ambiguity, we will assume that weight matrices remember which variables are involved in the monomial and the order they are written in. Since the concept of a weight matrix is merely an organizational tool, requiring this additional structure does not change the math involved.

The invariant subring R_n . For all positive integers n let

$$\begin{aligned} R_n &= (S^{\boxtimes n})^T \\ &= \left[\bigoplus_{d=0}^{\infty} (\text{Sym}^d V)^{\otimes n} \right]^T \end{aligned}$$

be the invariant subring generated by all invariant multi-homogeneous monomials. Here a multi-homogeneous monomial $f \in S^{\boxtimes n}$ of degree d is invariant (with respect to T) if and only if

$$\sum_{i=1}^n \sum_{j=1}^d |\phi_{ij}| = 0.$$

Equivalently, a multi-homogeneous monomial is invariant if and only if the weight matrix associated with it is zero-sum, i.e. its entries sum to zero.

Suppose that d_0 is a positive integer such that each R_n is generated by multi-homogeneous monomials in degrees $\leq d_0$. A proof that such a d_0 exists can be found in [1], which we modestly strengthen in §3.1. Our main theorem is:

Theorem 2.2. *For all positive integers n , we will let*

$$P_n = \text{Sym} \left(\bigoplus_{d=0}^{d_0} R_n^{(d)} \right).$$

Then exists positive integer d_1 such that the kernel \mathcal{I}_n of the natural surjection $P_n \rightarrow R_n$ is generated as a P_n -module in degrees $\leq d_1$ for all n .

3. PROOF OF THEOREM 2.2

Before we can prove Theorem 2.2, we would like to make some clarifications. First, we will clarify the difference between monomials and degrees in R_n and those in P_n . Second we will elaborate on what relations on R_n look like as elements of \mathcal{I}_n and as weight matrices.

The elements of P_n are polynomials with indeterminates in the set of multi-homogeneous monomials in R_n of degree $\leq d_0$. Because of this, there are two related notions of polynomials and degrees. Therefore, for the sake of clarity, multi-homogeneous monomials in R_n of degree $d \leq d_0$ will be called variables in P_n with *multi-homogeneous degree d* and will be denoted by the lowercase roman letters (e.g., f and g). Here the basis elements involved in these variables will be denoted with the greek letters (e.g., $\phi, \gamma \in \{x_1, \dots, x_k\}$). By contrast, the notion of monomials and degrees in P_n will have the usual, understood meaning inside of a polynomial ring. These monomials will be denoted by the capital roman letters (e.g., F and G). An example of this notation is seen below in (3).

Next note that \mathcal{I}_n is a toric variety. Then \mathcal{I}_n is generated by binomials. Consider a reduced binomial $F - G \in \mathcal{I}_n$ where

$$\begin{aligned} F &= f^1 \cdots f^p, \quad \text{where } f^\nu = f_1^\nu \otimes \cdots \otimes f_n^\nu \quad \text{and} \quad f_i^\nu = \phi_{i1}^\nu \phi_{i2}^\nu \cdots \\ G &= g^1 \cdots g^q, \quad \text{where } g^\mu = g_1^\mu \otimes \cdots \otimes g_n^\mu \quad \text{and} \quad g_i^\mu = \gamma_{i1}^\mu \gamma_{i2}^\mu \cdots \end{aligned} \tag{3}$$

for variables f^ν and g^μ in P_n . Note that if we forget the order of the basis elements in each f^ν and g^μ the products of the f^ν is the same in R_n as the product of the g^μ in terms of the basis elements. Therefore, for each fixed row of $\|F\|$, the entries

can be permuted to form that row of $\|G\|$. The width of these two weight matrices are the same and represent the total multi-homogeneous degree of the relation.

In order to prove Theorem 2.2, we will show we can factor this permutation into permutations of the matrix elements from a bounded number of columns, with each element permuted within its row. Each factor of this permutation represents a sub-relation where the degree of this sub relation is bounded by d_0 times the number of entries swapped. We'll factor the permutation by induction on the total multi-homogeneous degree of the relation.

Proof of Theorem 2.2. The first section §3.1 summarizes [1] with a slight change: we are finding a bound d_0 given by the set \mathcal{X}^2 rather than \mathcal{X} . We use this d_0 to find columns that form zero-sum sub-matrices for our relation. In §3.3 we find an analogous bound to find rows that form zero-sum sub-matrices for our relation.

In §3.2 we use our bound d_0 to reduce the problem to considering binomial relations $F - G$ where F and G respectively contain variable f and g of equal multi-homogeneous degree. We use this reduction in §3.4 we prove that each relation is generated in degrees $\leq n_0 d_0^2$.

3.1. It was proven in [1] that there exists a number D , dependent only on the representation (ρ, V) of T , such that given any zero-sum matrix with entries in \mathcal{X} , we can rearrange the matrix entries within their rows such that the Euclidean norm of all the column sums of the resulting matrix are bounded above by D .

Let \mathcal{A} be the set of all vectors in \mathbb{Z}^{2r} which are linear combinations of vectors in \mathcal{X}^2 and of Euclidean norm at most $2D$. This set is finite, so we can write $\mathcal{A} = \{a_1, \dots, a_L\}$, which defines a rational polyhedral cone

$$\sigma = \left\{ (\lambda_1, \dots, \lambda_L) \in \mathbb{R}^L \mid \sum_i \lambda_i a_i = 0, \text{ and each } \lambda_i \geq 0 \right\}.$$

Let $\Lambda = \sigma \cap \mathbb{Z}^L$. Then Λ is finitely generated as a semi-group (see Gordon's Lemma [2]). Fix a finite generating set S of Λ . Then define

$$d_0 = \max \{ \lambda_1 + \dots + \lambda_L \mid (\lambda_1, \dots, \lambda_L) \in S \}. \quad (4)$$

3.2. Fix a positive integer n and a reduced binomial relation $F - G \in \mathcal{J}_n$ as given in (3) such that the Euclidean norms of the column sums of $\|f^\nu\|$ and $\|g^\mu\|$ are bounded above by D given in §3.1, and so the the Euclidean norms of the column sums of $\|F\|$ and $\|G\|$ are bounded above by $2D$.

Let M be the matrix with entries in \mathcal{X}^2 whose ij -th entry is the concatenation of the ij -th entries of $\|F\|$ and $\|G\|$ respectively. The matrices $\|F\|$ and $\|G\|$ are both zero-sum, and so the Euclidean norms of the column sums of M are bounded above by D . Therefore, there exist $\leq d_0$ columns $\{j_1, \dots, j_d\}$ that form a zero-sum sub-matrix of M . Suppose that each j_ℓ -th column corresponds to the multi-homogeneous monomials $\phi_{1\nu_\ell}^{\nu_\ell} \otimes \dots \otimes \phi_{n\nu_\ell}^{\nu_\ell}$ and $\gamma_{1u_\ell}^{\mu_\ell} \otimes \dots \otimes \gamma_{nu_\ell}^{\mu_\ell}$ of degree 1 in $S^{\boxtimes n}$. Let

$$\begin{aligned} f &= f_1 \otimes \dots \otimes f_n, \quad \text{where } f_i = \phi_{iv_1}^{\nu_1} \dots \phi_{iv_d}^{\nu_d} \\ g &= g_1 \otimes \dots \otimes g_n, \quad \text{where } g_i = \gamma_{iu_1}^{\mu_1} \dots \gamma_{iu_d}^{\mu_d}. \end{aligned}$$

Note that f and g are both variables in P_n with equal multi-homogeneous degree. Next let $\mathcal{V} = \{\nu_1, \dots, \nu_d\}$ and $\mathcal{U} = \{\mu_1, \dots, \mu_d\}$ and

$$F_1 = \prod_{\nu \in \mathcal{V}} f^\nu \quad \text{and} \quad F_2 = \prod_{\mu \in \mathcal{U}} g^\mu.$$

Here we do not take the product over all ν_ℓ and μ_ℓ respectively because there may be repeated indices (i.e., $\nu_\ell = \nu_{\ell'}$) and we want F' and G' to be factors of F and G respectively. Note that F' and G' are monomials in P_n in degree $\leq d \leq d_0$. Therefore, we have binomial relations

$$F_1 - f \cdot G_1 \quad \text{and} \quad F_2 - g \cdot G_2$$

in \mathcal{J}_n in degrees $\leq d_0^2$ for appropriate monomials G_1 and G_2 in P_n . Furthermore, we have the decomposition.

$$F - G = H_1(F_1 - f \cdot G_1) + (f \cdot H_1 \cdot G_1 - g \cdot H_2 \cdot G_2) + H_2(F_2 - g \cdot G_2) \in \mathcal{J}_n$$

for appropriate monomials H_1 and H_2 in P_n . Therefore, $f \cdot H_1 \cdot G_1 - g \cdot H_2 \cdot G_2$ is a binomial relation. Thus the problem is reduced to the case of binomial relations such that there is a multi-homogeneous degree d variable in each term for some $d \leq d_0$.

3.3. Let \mathcal{B} be the subset of vectors in \mathbb{Z}^{2r} which are the linear combinations in of vectors in \mathcal{X}^2 such that the sum of the coefficients is at most d_0 . Using the analogous method described in §3.1 for \mathcal{B} (analogous to \mathcal{A}), we define n_0 (analogous to d_0).

3.4. Fix a positive integer n . Let $F - G \in \mathcal{J}_n$ be a reduced binomial relation such as that given in (3), but now assume that both $f = f^1$ and $g = g^1$ have multi-homogeneous degree $d \leq d_0$; define matrix M as in §3.2. By §3.3 we can partition $\{I_1, \dots, I_L\}$ of rows $[n]$ such that each block of rows is size $\leq n_0$ and forms a zero-sum sub-matrix of M . Suppose that each block I_ℓ corresponds the pure tensors

$$a_{I_\ell} = a_1 \otimes \dots \otimes a_n, \quad \text{where } a_i = \begin{cases} 1 & \text{if } i \notin I_\ell \\ \phi_{i1} \dots \phi_{id} & \text{if } i \in I_\ell \end{cases}$$

$$b_{I_\ell} = b_1 \otimes \dots \otimes b_n, \quad \text{where } b_i = \begin{cases} 1 & \text{if } i \notin I_\ell \\ \gamma_{i1} \dots \gamma_{id} & \text{if } i \in I_\ell \end{cases}$$

Let $m_0 = a_{I_1} \dots a_{I_L} = f$. Then for all $\ell \in [L]$ recursively let

$$m_\ell = \frac{m_{\ell-1}}{a_{I_\ell}} b_{I_\ell}.$$

Note that each m_ℓ is a variable in P_n of multi-homogeneous degree d since the weight matrix of m_ℓ is the weight matrix of $m_{\ell-1}$ with a zero-sum block of rows replaced by another zero-sum block of rows, and hence itself zero-sum. Also notice that this sequence terminates at $m_L = b_{I_1} \dots b_{I_L} = g$. Since we are swapping at most $n_0 d_0$ basis elements for each m_ℓ , the sub-relation that represents the swapping

$$m_{\ell-1} \cdot F_\ell - m_\ell \cdot G_\ell \in \mathcal{J}_n,$$

for appropriate monomials F_ℓ and G_ℓ , has at most degree $n_0 d_0^2$. Furthermore, we have the decomposition

$$F - G = H_1(f \cdot F_1 - m_1 \cdot G_1) + \dots + H_L(m_{L-1} \cdot F_L - g \cdot G_L) + (g \cdot H_L \cdot F_L - G) \in \mathcal{J}_n$$

For appropriate monomials H_ℓ . Since that last sub-relation can have g factored out of both terms, we have reduced the total multi-homogeneous degree of the binomial relation. Then by induction on the total multi-homogeneous degree on relations,

that last term is generated in degrees $\leq n_0 d_0^2$. Hence we have proved that $F - G$ can be generated in degrees $\leq n_0 d_0^2$. \square

REFERENCES

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- [3] Andrew Snowden, *Finiteness Conjectures for Quotients of n Points on a Variety*.